MECHANICS OF THE GEOMAGNETIC DYNAMO

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Mechanics of the Geomagnetic Dynamo

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Abstract

After developing the formal integration of $\partial \underline{B}/\partial t = \nabla \times (\underline{v} \times \underline{B})$, it is shown that cyclonic convective motions in the core produce magnetic loops in meridional planes through interaction with the toroidal magnetic field. Expressing these loops in terms of the usual orthogonal vector modes, it is shown that they result in a predominantly dipole field. Together with the nonuniform rotation of the core, which produces the toroidal field from the dipole field, the cyclonic motions result in a complete self-regenerating magnetic dynamo. We conclude that any rotating, convecting, electrically conducting body of sufficient size will possess a magnetic field generated by this dynamo mechanism. The possibility of an abrupt reversal of the field is discussed.

1. Introduction

In the magnetohydrodynamic theory of the Earth's magnetic field (Elsasser, 1946; Bullard, 1949), it is shown that the nonuniform rotation of the liquid core of the Earth will produce a strong toroidal field, (that is, a field along the circles of latitude). The nonuniform rotation of the core is the result of the Coriolis forces which act upon the rising and falling convective motions. If we are to show that the dipole field is due to a self-sustaining dynamo, we must demonstrate the existence of a feedback link, i.e. of the mechanism by which the dipole field is regenerated from the toroidal field.

Cowling (1933) has shown that there is no very direct way of producing a dipole field with simple fluid motion; we must in particular abandon all models possessing rotational symmetry. Cowling's point is that in a dipole field there must be a line singularity about which the magnetic field circles. The field vanishes at the singularity, but the curl does not. But for a steady state field, the field equations tell us that the field $\mathbf{B}$, curl $\mathbf{B}$, and the current density $\mathbf{i}$, must all become small of the same order if one of these quantities is small. Hence such a field cannot be maintained in a steady state. The linearity in $\mathbf{B}$ implies, furthermore, that a time dependent velocity field is equally unable to maintain or amplify the field in the average.

In this paper we first point out that, owing to the Coriolis force, the radial convective streams must produce local cyclonic and anticyclonic circulations somewhat resembling those observed in the atmosphere. The deformation of the toroidal field by these motions can be shown to give rise to the formation of loops of the magnetic lines of force in the meridional planes which coalesce to a mean meridional field. The dipole can be shown to be by far the largest harmonic component of this field. Thus the interaction of the cyclonic and anticyclonic local motions with the toroidal field provides the desired feedback link; together with the interaction of the nonuniform rotation of the core and the dipole field, this constitutes a complete regenerative cycle.

2. Dynamics

The primary motions in the core are assumed to be convective motions in a radial direction, (Bullard, 1949; Elsasser, 1950). It is readily demonstrated that they result in a nonuniform rotation of the core in which the outer layers of the core lag behind the average rotation and the inner ones exceed the average.

In the atmosphere of the Earth the Coriolis force of the large scale cyclonic motions shears the vertical motion to such an extent that the actual direction of flow in a rising eddy may be less than a degree from the horizontal. We do not expect the shearing to be so large in the core because of the retarding forces exerted by the toroidal field. Bullard (1954) has shown that for any self-sustaining dynamo there exists a steady state velocity which, when exceeded, causes the magnetic field to grow until its reactive forces balance the mechanical forces driving the dynamo. Assuming that the core of the Earth is a self-sustaining dynamo, we expect to find it operating near the steady state velocity with the Coriolis force balancing the reaction of the toroidal field.

Defining the magnetic Reynolds number of the nonuniform rotation in terms of the radius of the core, Bullard (1954) finds it to be of the order of 25 for steady state operation. If we choose $\sigma = 3.5 \times 10^4$ mho/m (Elsasser, 1950) we obtain toroidal velocities of the order of 0.1 mm/sec. To obtain an estimate of the radial velocities required we shall assume a suitable value for the toroidal field and estimate the Coriolis force needed to overcome its reaction. A value of 40 gauss seems a reasonable lower limit on the toroidal field, B_φ, and four gauss for the dipole field B_d. The reactive force per unit mass is $(\nabla \times \mathbf{B}) \times \mathbf{B}/(\mu\rho)$ newtons/kg where ρ is the density of the medium. Approximating this as $B_\varphi B_d/(L\mu\rho)$ where L is some characteristic length, we obtain a force of the order of 1.3×10^{-10} knewtons/kg for $\rho = 10$ gm/cm^3 and $L = 1000$ km. With $\omega \sim 10^{-4}$ sec^{-1} we obtain a lower limit on the average radial velocity of 10^{-3} mm/sec. The individual active convective regions probably have radial velocities one or two powers of ten larger than this, say 10^{-2} to 10^{-1} mm/sec, and we estimate the ratio of toroidal to radial velocity to be ten or less, whereas in the atmosphere this ratio is generally of the order of a hundred.

In this paper we shall disregard the nonuniform rotation and shall consider only the interaction of the radial motions with the toroidal field. Consider, then, a radial, convective stream with the associated influx and efflux of matter at its ends. A rising stream will result in an efflux of fluid near the surface of the core and an influx near the center, and conversely for a descending stream. As is well known from meteorological phenomena, the Coriolis forces resulting from the influx and efflux produce a rotation of the convective column about its axis. To formulate this, we define a local cartesian coordinate system (ξ, η, ζ) where the ξ-axis is tangent to the line of longitude through the origin and the positive ξ-axis points south, the η-axis is tangent to a line of latitude through the origin and points east, and the ζ-axis is directed radially outwards. We define

$$\rho = \sqrt{(\xi^2 + \eta^2)}$$

and the polar angle ψ measuring azimuth about the ζ-axis from the ξ-axis. In this system we represent an outflow from the radial motion as

$$\underset{\sim}{v}_1 = v_1(\rho)\left(\underset{\sim}{e}_\xi \cos\psi + \underset{\sim}{e}_\eta \sin\psi\right), \quad \sin\psi = \eta/\rho, \quad \cos\psi = \xi/\rho$$

where the $\underset{\sim}{e}$'s are unit vectors in the corresponding directions. The angular velocity of the Earth is

$$\underset{\sim}{\omega} = \omega\left(\underset{\sim}{e}_\zeta \cos\theta - e_\xi \sin\theta\right)$$

where θ is the colatitude of the origin of the (ξ, η, ζ) system. The Coriolis force due to this efflux is

$$-2(\underset{\sim}{\omega} \times \underset{\sim}{v}_1) = 2\omega v_1(\rho)\left[e_\xi \cos\theta \cos\psi - e_\eta \cos\theta \cos\psi - e_\zeta \sin\theta \sin\psi\right]$$

The ζ-component of the curl of this force is

$$\operatorname{curl}_\zeta\left[-2(\underset{\sim}{\omega} \times \underset{\sim}{v}_1)\right] = \frac{\partial}{\partial\xi}\left(-2\underset{\sim}{\omega} \times \underset{\sim}{v}_1\right)_\eta - \frac{\partial}{\partial\eta}\left(-2\underset{\sim}{\omega} \times \underset{\sim}{v}_1\right)_\xi$$

$$= -2\omega\cos\theta \frac{1}{\rho}\frac{d}{d\rho}(\rho v_1)$$

$d(\rho v_1)/d\rho$ is positive for an efflux and negative for an influx; this defines the sense of the circulation.

We shall furthermore assume that the dissipative forces of viscosity, turbulence, etc. may be neglected in a first approximation (the Reynolds number being large) with the result that the convecting regions will always show a sense of circulation corresponding to an <u>influx</u>: The convective region is set in rotation by the influx; in the absence of dissipation the angular momentum of the convective column is conserved and is stopped only by the equal and opposite angular impulse of the efflux at the terminus of the convective stream. In the northern hemisphere the circulation is counterclockwise, in the southern hemisphere clockwise, if viewed from outside the core. These motions are illustrated in figure 1.

3. Magnetic Field

Bullard (1954) has analyzed the toroidal magnetic field generated by the interaction of the nonuniform rotation with the dipole field. The result essential to the feedback link discussed in this paper is that the toroidal field has a maximum about half or two thirds of the way out from the center of the core and drops off to zero near the center and near the surface.

The conductivity of the core is sufficiently high so that fluid velocities of 10^{-2} mm/sec or more may be expected to carry the lines of force along bodily. Thus, a convective upwelling produces an upward bulge in the toroidal field. The rotation of the upwelling about the radial direction twists this bulge into a loop with a nonvanishing projection on the meridional plane resulting, in the northern hemisphere, in the sequence illustrated in figure 2. Figure 3 shows the similar process occurring in a sinking column of fluid in the northern hemisphere. In the southern hemisphere the sense of the toroidal field and of the rotation of the convective column are reversed. Figure 4 illustrates the complete dynamo model: A typical line of force of the dipole field of the Earth is shown as having been drawn out by the nonuniform rotation to give a toroidal field; the loops produced from the toroidal field by convective motions in both hemispheres are shown. We note that the loops produced nearest the surface of the core arise from rising convective currents and have the same sense of circulation as the overall external dipole field of the core; the loops produced near the center of the core have the opposite sense, but their great depth means that they will decay before diffusing to the upper part of the core where the main amplification of the toroidal field occurs. The dipole field observed outside the core, which is due to the coalescence of all the loops formed by the convective motions, will have the same sense as the original dipole field.

This qualitative discussion indicates that the simple dynamical model of the convective motions presented in the previous section should result in a self-sustaining dynamo with a dipole field*. Insofar as our model is correct, it would follow that any sufficiently large, convecting, electrically conducting fluid body should exhibit a magnetic dipole roughly parallel to the axis of rotation if the body rotates rapidly enough. On reversing the sense of either the magnetic field or the rotation of the body in the diagrams we see that the amplificatory process is independent of either. Thus, the sense of the dipole field will depend only on the sense of the initial field from which it was generated.

*The entire process has been dubbed the bathtub <u>effect</u> for, as every physics freshman has heard, the rotation of the water running into the drain of a bathtub is the result of the Coriolis force.

4. Generation of Loops

The basic equations for the deformation of the magnetic field $\underset{\sim}{B}$ by the velocity field $\underset{\sim}{v}$ are, for a perfectly conducting fluid (Elsasser, 1950)

$$\frac{\partial \underset{\sim}{B}}{\partial t} = \nabla \times (\underset{\sim}{v} \times \underset{\sim}{B}) \tag{1}$$

This differential equation is equivalent to the integral equation

$$\frac{d}{dt} \int \underset{\sim}{B} \cdot d\underset{\sim}{S} = 0 \tag{2}$$

where $d\underset{\sim}{S}$ is an element of a surface moving with the fluid. It follows that the complete integral of (1) is

$$\int \underset{\sim}{B} \cdot d\underset{\sim}{S} = \text{constant} \tag{3}$$

which may be written as three linear simultaneous algebraic equations for B_i by choosing infinitesimal areas over which to integrate. Thus we consider the infinitesimal element of volume determined at time $t = 0$ by the three linearly independent, but otherwise arbitrary, vectors δa_i, δb_i, δc_i. Let now one corner of this element of volume be at x_0^i, then

$$\delta a_i = x_1^i - x_0^i \; , \quad \delta b_i = x_2^i - x_0^i \; , \quad \delta c_i = x_3^i - x_0^i \tag{4}$$

At some later time, t, we write for the same material element

$$\delta A_i = X_1^i - X_0^i \; , \quad \delta B_i = X_2^i - X_0^i \; , \quad \delta C_i = X_3^i - X_0^i \tag{5}$$

The Lagrangian coordinates X^i are functions of the initial position x^i and of t, so that

$$X^i = X^i(x^i, t) \; .$$

Now

$$X_1^{\,i} = X^i(x_\circ^{\,j},t) = X^i(x_\circ^{\,j}+\delta a_j,t) = X^i(x_\circ^{\,j},t) + \frac{\partial X^i(x_\circ^{\,j},t)}{\partial x_\circ^{\,k}}\,\delta a_k + O^2(\delta a_i)$$

and similarly for $X_2^{\,i}$ and $X_3^{\,i}$. Hence, by (5)

$$\delta A_i = \frac{\partial X^i(x_\circ^{\,j},t)}{\partial x_\circ^{\,k}}\,\delta a_k\,,\ \delta B_i = \frac{\partial X^i(x_\circ^{\,j},t)}{\partial x_\circ^{\,k}}\,\delta b_k\,,\ \delta C_i = \frac{\partial X^i(x_\circ^{\,j},t)}{\partial x_\circ^{\,k}}\,\delta c_k \tag{6}$$

The areas of the faces of the element of volume are (we use summation convention)

$$\epsilon^{ijk}\,\delta A_j\,\delta B_k\,,\qquad \epsilon^{ijk}\,\delta B_j\,\delta C_k\,,\qquad \epsilon^{ijk}\,\delta C_j\,\delta A_k$$

where ϵ^{ijk} is the usual permutation tensor, being +1 if ijk is an even, -1 if it is an odd permutation of 1, 2, 3, and zero otherwise. If the magnetic field is written as $B_i(X^j,t)$ with the initial value $B_i(x^j,o) = b_i(x^j)$, the flux through the face bounded by δA_i and δB_i is initially $b_i\,\epsilon^{ijk}\,\delta A_j\,\delta B_k$; at time t it is $B_i\,\epsilon^{ijk}\,\delta A_j\,\delta B_k$, and similarly for the other two sides. Thus, upon using (6) to eliminate δA_i, δB_i and δC_i, we find from (3)

$$\begin{aligned}
b_i\,\epsilon^{ijk}\,\delta a_j\,\delta b_k &= B_i\,\epsilon^{ijk}\frac{\partial X^j}{\partial x^r}\frac{\partial X^k}{\partial x^s}\,\delta a_r\,\delta b_s\\
b_i\,\epsilon^{ijk}\,\delta b_j\,\delta c_k &= B_i\,\epsilon^{ijk}\frac{\partial X^j}{\partial x^r}\frac{\partial X^k}{\partial x^s}\,\delta b_r\,\delta c_s\\
b_i\,\epsilon^{ijk}\,\delta c_j\,\delta a_k &= B_i\,\epsilon^{ijk}\frac{\partial X^j}{\partial x^r}\frac{\partial X^k}{\partial x^s}\,\delta c_r\,\delta a_s
\end{aligned} \tag{7}$$

Since the δa_i, δb_i, and δc_i are essentially arbitrary, we may set

$$\delta a_i = \delta_{1i}\,,\qquad \delta b_i = \delta_{2i}\,,\qquad \delta c_i = \delta_{3i}$$

where δ_{ij} is the usual Kronecker delta. Then (7) reduces to

$$b_3 = B_i\,\epsilon^{ijk}\frac{\partial X^j}{\partial x^1}\frac{\partial X^k}{\partial x^2}\,,\quad b_1 = B_i\,\epsilon^{ijk}\frac{\partial X^j}{\partial x^2}\frac{\partial X^k}{\partial x^3}\,,\quad b_2 = B_i\,\epsilon^{ijk}\frac{\partial X^j}{\partial x^3}\frac{\partial X^k}{\partial x^1} \tag{8}$$

These three linear algebraic equations can be solved to yield $B_i(X^j,t)$ in terms of $b_i(x^j)$ and the Lagrangian coordinates X^j. To facilitate this solution we denote by J_{ij} the derivative $\partial X^i/\partial x^j$, the J_{ij} form a matrix which we shall write as (J). The determinant of this matrix, denoted by $|J|$, is the Jacobian of the transformation $X^i = X^i(x^j, t)$. If (b) and (B) denote the column matrices with components b_i and B_i respectively, then (28) may be written

$$(b) = (\mathrm{adj}\, J)(B)$$

where the adjoint matrix is defined as usual, by the relation

$$\mathrm{adj}\, J = (J)^{-1}\, |J|$$

and hence

$$(B) = \frac{1}{|J|}\,(J)(b) \tag{9}$$

Thus, finally[1)]

$$B_i = \frac{1}{|J|}\, b_j \frac{\partial X^i}{\partial x^j} \tag{10}$$

which is the general solution of (1).

We shall now use (10) to demonstrate that rotating convective currents will produce loops of magnetic flux which are displaced, relative to the height of the toroidal field, in the direction of the convective motion. Since we use the Lagrangian method, the velocity field must be expressed in terms of the displacement of the individual fluid particles. On the assumptions made the trajectory of a particle will, in the first approximation, be a helix. We set

$$\frac{d\xi}{dt} = -\frac{w_o}{a}\,\eta\, S(\rho)\,,\quad \frac{d\eta}{dt} = \frac{w_o}{a}\,\xi\, S(\rho)\,,\quad \frac{d\zeta}{dt} = v_o\, R(\rho) \tag{11}$$

The pitch of each helix is $(v_o/w_o)(a/\rho)\, R(\rho)/S(\rho)$

1) Some steps of this integration procedure have already been indicated by Lundquist (1952)

Integrating (11) we obtain

$$\rho = \lambda,\ \xi = \lambda\cos\left[\tfrac{w_0}{a} S(\lambda)t + \lambda_2\right],\ \eta = \lambda\sin\left[\tfrac{w_0}{a} S(\lambda)t + \lambda_2\right],\ \zeta = v_0 R(\lambda)t + \lambda_1 \qquad (12)$$

where λ, λ_1 , and λ_2 are constants of integration. If $\xi = \xi_0$, $\eta = \eta_0$ and $\zeta = \zeta_0$ for $t = 0$, then

$$\lambda = \sqrt{(\xi_0^2 + \eta_0^2)}\ ,\ \lambda_1 = \zeta_0\ ,\ \cot\lambda_2 = \frac{\xi_0}{\eta_0} \qquad (13)$$

It follows that

$$\frac{\partial\lambda}{\partial\xi_0} = \frac{\xi_0}{\lambda},\ \frac{\partial\lambda}{\partial\eta_0} = \frac{\eta_0}{\lambda},\ \frac{\partial\lambda}{\partial\zeta_0} = 0;\ \frac{\partial\lambda_1}{\partial\xi_0} = \frac{\partial\lambda_1}{\partial\eta_0} = 0,\ \frac{\partial\lambda_1}{\partial\zeta_0} = 1$$
$$\frac{\partial\lambda_2}{\partial\xi_0} = -\frac{\eta_0}{\lambda^2},\ \frac{\partial\lambda_2}{\partial\eta_0} = \frac{\xi_0}{\lambda^2},\ \frac{\partial\lambda_2}{\partial\zeta_0} = 0 \qquad (14)$$

Using (14) it is readily shown from (12) that

$$\frac{\partial\xi}{\partial\xi_0} = \frac{1}{\lambda^2}(\xi_0\xi + \eta_0\eta) - \frac{w_0}{a}\xi_0 S'(\lambda)t\frac{\eta}{\lambda}$$

$$\frac{\partial\xi}{\partial\eta_0} = \frac{1}{\lambda^2}(\eta_0\xi - \xi_0\eta) - \frac{w_0}{a}\eta_0 S'(\lambda)t\frac{\eta}{\lambda}\ ,\quad \frac{\partial\xi}{\partial\zeta_0} = 0.$$

$$\frac{\partial\eta}{\partial\xi_0} = -\frac{1}{\lambda^2}(\eta_0\xi - \xi_0\eta) + \frac{w_0}{a}\xi_0 S'(\lambda)t\frac{\xi}{\lambda}$$

$$\frac{\partial\eta}{\partial\eta_0} = \frac{1}{\lambda^2}(\xi_0\xi + \eta_0\eta) + \frac{w_0}{a}\eta_0 S'(\lambda)t\frac{\xi}{\lambda}\ ,\quad \frac{\partial\eta}{\partial\zeta_0} = 0$$

$$\frac{\partial\zeta}{\partial\xi_0} = v_0 R'(\lambda)t\frac{\xi_0}{\lambda}\ ,\quad \frac{\partial\zeta}{\partial\eta_0} = v_0 R'(\lambda)t\frac{\eta_0}{\lambda}\ ,\quad \frac{\partial\zeta}{\partial\zeta_0} = 1$$

The flow is incompressible and $|J| = 1$.

We now specify our magnetic field as purely toroidal and locally a function of height only, say

$$B_\xi = B_\zeta = 0\ ,\ B_\eta = B(\zeta) \qquad (15)$$

From (10) we then find

$$B_\xi = B(\zeta_0)\frac{\partial \xi}{\partial \eta_0} = B(\zeta_0)\left[\frac{1}{\lambda^2}\left(\eta_0 \xi - \xi_0 \eta\right) - \frac{w_0}{a}\eta_0\, S'(\lambda) t\, \frac{\eta}{\lambda}\right]$$

$$B_\eta = B(\zeta_0)\frac{\partial \eta}{\partial \eta_0} = B(\zeta_0)\left[\frac{1}{\lambda^2}\left(\xi_0 \xi + \eta_0 \eta\right) + \frac{w_0}{a}\eta_0\, S'(\lambda) t\, \frac{\xi}{\lambda}\right] \qquad (16)$$

$$B_\zeta = B(\zeta_0)\frac{\partial \zeta}{\partial \eta_0} = B(\zeta_0)\, v_0\, R'(\lambda)\, t\, \frac{\eta_0}{\lambda}$$

Let us now use (12) to eliminate ξ_0, η_0, and ζ_0 from (16). From (13) we may rewrite (12) as

$$\xi = \xi_0 \cos\left[\frac{w_0}{a} S(\lambda) t\right] - \eta_0 \sin\left[\frac{w_0}{a} S(\lambda) t\right]$$

$$\eta = \xi_0 \sin\left[\frac{w_0}{a} S(\lambda) t\right] + \eta_0 \cos\left[\frac{w_0}{a} S(\lambda) t\right]$$

Solving these two equations for ξ_0 and η_0 and using (12) again we obtain

$$\xi_0 = \xi \cos\left[\frac{w_0}{a} S(\lambda) t\right] + \eta \sin\left[\frac{w_0}{a} S(\lambda) t\right]$$

$$\eta_0 = -\xi \sin\left[\frac{w_0}{a} S(\lambda) t\right] + \eta \cos\left[\frac{w_0}{a} S(\lambda) t\right]$$

$$\zeta_0 = \zeta - v_0 R(\lambda) t \qquad (17)$$

We let $\xi = \lambda \cos\psi$ and $\eta = \lambda \sin\psi$ so that

$$\xi_0 = \lambda \cos\left[\psi - \frac{w_0}{a} S(\lambda) t\right], \quad \eta_0 = \lambda \sin\left[\psi - \frac{w_0}{a} S(\lambda) t\right] \qquad (18)$$

Using (17) and (18) we now rewrite (16) as

$$\left.\begin{aligned}
B_\xi &= -B(\zeta - v_0 R(\lambda) t)\left\{\sin\left(\frac{w_0}{a} S(\lambda) t\right) + \frac{w_0}{a} S'(\lambda)\,\lambda t \sin\psi \sin\left[\psi - \frac{w_0}{a} S(\lambda) t\right]\right\} \\
B_\eta &= +B(\zeta - v_0 R(\lambda) t)\left\{\cos\left(\frac{w_0}{a} S(\lambda) t\right) + \frac{w_0}{a} S'(\lambda)\,\lambda t \cos\psi \sin\left[\psi - \frac{w_0}{a} S(\lambda) t\right]\right\} \\
B_\zeta &= +B(\zeta - v_0 R(\lambda) t)\, v_0 R'(\lambda) t \sin\left[\psi - \frac{w_0}{a} S(\lambda) t\right]
\end{aligned}\right\} \qquad (19)$$

Let us investigate the field in the $\zeta\xi$ - plane. We put $\psi = 0$ and obtain

$$\left.\begin{aligned} B_\xi &= -B(\zeta - v_0 R(\lambda)t)\sin\left[\frac{w_0}{a}S(\lambda)t\right] \\ B_\eta &= +B(\zeta - v_0 R(\lambda)t)\left\{\cos\left[\frac{w_0}{a}S(\lambda)t\right] - \frac{w_0}{a}S'(\lambda)\lambda t\sin\left[\frac{w_0}{a}S(\lambda)t\right]\right\} \\ B_\zeta &= -B(\zeta - v_0 R(\lambda)t)\, v_0 R'(\lambda)t\sin\left[\frac{w_0}{a}S(\lambda)t\right] \end{aligned}\right\} \quad (20)$$

We are primarily interested in the loops formed and we therefore subtract from this field the portions that do not contribute to the loops, namely, those due to the motion in the ζ direction alone and those due to the circular motion about the ζ - axis alone, so as to retain only the part that results from the superposition of these motions. Thus, for $w_0 = 0$ we write the field as

$$\alpha_\xi = 0\ ,\quad \alpha_\eta = B(\zeta - v_0 R(\lambda)t)\ ,\quad \alpha_\zeta = 0$$

For $v_0 = 0$ we write

$$\beta_\xi = -B(\zeta)\sin\left[\frac{w_0}{a}S(\lambda)t\right]$$

$$\beta_\eta = +B(\zeta)\left\{\cos\left[\frac{w_0}{a}S(\lambda)t\right] - \frac{w_0}{a}S'(\lambda)\lambda t\sin\left[\frac{w_0}{a}S(\lambda)t\right]\right\}$$

$$\beta_\zeta = 0$$

Hence

$$B_\xi - \alpha_\xi - \beta_\xi = \left[B(\zeta) - B(\zeta - v_0 R(\lambda)t)\right]\sin\left[\frac{w_0}{a}S(\lambda)t\right]$$

$$B_\eta - \alpha_\eta - \beta_\eta = \left[B(\zeta - v_0 R(\lambda)t) - B(\zeta)\right]\left\{\cos\left[\frac{w_0}{a}S(\lambda)t\right] - \frac{w_0}{a}S'(\lambda)\lambda t\sin\left[\frac{w_0}{a}S(\lambda)t\right]\right\}$$
$$- B(\zeta - v_0 R(\lambda)t)$$

$$B_\zeta - \alpha_\zeta - \beta_\zeta = -B(\zeta - v_0 R(\lambda)t)\, v_0 R'(\lambda)t\sin\left[\frac{w_0}{a}S(\lambda)t\right]$$

The projections of the lines of force on the $\zeta\xi$ -plane satisfy the differential equation

$$\frac{d\zeta}{d\lambda} = \frac{B_\zeta - \alpha_\zeta - \beta_\zeta}{B_\xi - \alpha_\xi - \beta_\xi} = \frac{B(\zeta - v_0 R(\lambda)t)\, v_0 R'(\lambda)t}{B(\zeta - v_0 R(\lambda)t) - B(\zeta)} \tag{21}$$

We define the variable μ as

$$\mu \equiv v_0 R(\lambda) t \tag{22}$$

so that

$$d\mu = v_0 R'(\lambda)\, t\, d\lambda$$

Then

$$d\mu = \left[1 - \frac{B(\zeta)}{B(\zeta-\mu)}\right] d\zeta$$

This may be rewritten as

$$B(\zeta-\mu)\, d(\zeta-\mu) = B(\zeta)\, d\zeta \tag{23}$$

We define the function $I(x)$ as

$$I(x) = \int_0^x dx\; B(x) \tag{24}$$

The integral of (23) becomes

$$I(\zeta - v_0 R(\lambda) t) = I(\zeta) + C \tag{25}$$

If we assume that $R(\lambda)$ is a function having a maximum at $\lambda = 0$ and decreasing monotonically to zero for large $|\lambda|$, it can be shown that (25) results in magnetic loops; the loops for special case that

$$R(\lambda) = \exp(-\lambda^2/a^2)\,, \quad B(\zeta) = B_0 \exp(-\zeta^2/a^2)$$

are shown in figure 5 for $v_0 t = a$. The neutral point is $\xi = 0$, $\zeta = v_0 t/2 = a/2$.

Thus, we have demonstrated that the loops are displaced in the direction of the convective motion.

It is well to note that if w. is so large that the convective column rotates more than one half a revolution after passing through the toroidal field, the sense of part of each loop will be reversed. We shall assume that loops in the first half revolution predominate and the average sense is as illustrated in figures 2, 3, and 4.

In the next section where we study the coalescence of loops, we shall see that a set of loops of one sense of flux above the toroidal field and another set of the opposite sense below this field combine in such a way that the net average field is dominated by the loops at the higher level, i.e., the resulting poloidal field of the core has the same sense as the upper loops. We know little about the dynamics of convective motion and it is quite possible that in the core there are rising and descending streams in equal numbers and of symmetrical structure. Again, it is possible that the convective motion is assymmetrical, the ascending streams being concentrated in narrow regions and the compensating descending motions being spread over a large area (the familiar pattern of atmospheric thunder storms). Now it can be demonstrated that in such a model of convection only the intense rising currents produce appreciable magnetic loops whereas the contribution of the spread-out subsiding motions to the meridional magnetic field is negligible. This might be shown on using the preceding analysis, but the calculations turn out to be extremely involved. In the appendix we therefore solve equation (1) again, this time by a perturbation procedure. In the approximation which we use, the displacement of the loops in the direction of the convective stream does not yet appear, on the other hand we obtain simple analytical expressions for the secondary magnetic field, and these exhibit clearly the fact that the contribution of the spread-out descending motions to the meridional loops is negligible. The general conclusion is that we can construct a self-sustaining dynamo on the basis of either a symmetrical or an asymmetrical kinematical model of convection.

5. Coalescence of Loops

It remains to be shown that a number of loops with the sense indicated in figure 4 will actually result in a field observed outside the core which is predominantly a dipole field. The study of the geomagnetic secular variation shows (Elsasser, 1950) that there are about fourteen distinct local irregularities in the field of the Earth. Taking this figure as a lower limit on the total number of loops in various stages of decay in the core, the total number of loops might be as high as twice this amount. We shall assume that the loops are distributed randomly over a sphere of given radius concentric with the core, and we shall represent this distribution by a uniform average density.

Before we carry out the calculations, let us briefly discuss this averaging process. We shall find that the loops produced beneath the toroidal field by descending currents of fluid (and which have the "wrong" sense for feedback) are buried so deeply in the core that they do not contribute appreciably to the resultant mean poloidal field. Since the coalescence problem is linear, we may demonstrate this fact in the following fashion: We first compute the magnitude of the mean poloidal field with the restriction that the loops are confined within a sphere of arbitrary radius not exceeding the radius of the core. We can then represent the condition where there are loops of one sense in the upper and loops of the opposite sense in the lower part of the core by means of a linear superposition of two concentric spheres of different diameters, each being filled with loops of only one sign.

Let $r, \vartheta^*, \varphi^*$ be a system of polar coordinates whose origin is at the center of the core. Let a designate the linear dimensions of a typical loop; we assume the loops small, $a \ll P$ where P designates now the radius of the core. Consider a loop whose center is at $r = R, \vartheta^* = \Theta, \varphi^* = \Phi$. For any point on the loop we set

$$r = R + \rho, \quad \vartheta^* = \Theta + \vartheta, \quad \varphi^* = \Phi + \varphi \tag{26}$$

where ρ, ϑ, φ are small quantities of order a/R whose higher powers will be neglected.

To simplify the calculations we next assume that the loop can be described by means of a gaussian symmetrical about the loop center. We choose

$$B_\rho = -B_0 \frac{R}{a} \vartheta \exp\left[-\frac{1}{a^2}\left(\rho^2 + R^2\vartheta^2 + R^2 \sin^2\Theta\, \varphi^2\right)\right]$$

$$B_\vartheta = +B_0 \frac{\rho}{a} \exp\left[-\frac{1}{a^2}\left(\rho^2 + R^2\vartheta^2 + R^2 \sin^2\Theta\, \varphi^2\right)\right]$$

$$B_\varphi = 0 \,;$$

the lines of force are circles with centers on the φ-axis; together they form a torus of magnetic flux. Some of the later analysis is more conveniently carried out if we work with the curl of the magnetic field, since this is a toroidal rather than a poloidal vector (Elsasser 1946). We shall need only the φ-component of the curl for reasons that will appear presently.

$$(\nabla \times \mathbf{B})_\varphi = \frac{2B_0}{a}\left(1 - \frac{\rho^2}{a^2} - \frac{R^2}{a^2}\vartheta^2\right) \exp\left[-\frac{1}{a^2}\left(\rho^2 + R^2\vartheta^2 + R^2 \sin^2\Theta\, \varphi^2\right)\right] \tag{27}$$

We now expand (27) in terms of orthogonal toroidal vector modes, whereupon we average over all possible positions of the loop centers. If the distribution function of the loops is independent of Φ, only the rotationally symmetric modes (zonal harmonics) will survive the averaging process. The toroidal zonal modes are

$$T_{\rho(ns)} = T_{\vartheta(ns)} = 0\,, \quad T_{\varphi(ns)} = c_{ns}\, j_n(k_{sn} r)\, \frac{dP_n(\cos\vartheta^*)}{d\vartheta^*} \tag{28}$$

where $j_n(x)$ is a spherical Bessel function defined (Stratton, 1941) as $(\pi/(2x))^{1/2} J_{n+1/2}(x)$ and where the boundary conditions require the k_{sn} to be the roots of (Elsasser, 1946)

$$j_{n-1}(k_{sn} P) = 0, \tag{29}$$

P being the radius of the core.

We now develop (28) in ascending powers of the small quantities ρ and ϑ defined in (26).

$$T_{\varphi(ns)} = c_{ns}\left(A_{ns} + B_{ns}\rho + C_{ns}\vartheta + D_{ns}\rho^2 + E_{ns}\rho\vartheta + F_{ns}\vartheta^2 + O^3(\rho,\vartheta)\right) \tag{30}$$

We will be interested only in D_{ns} and F_{ns}, which are readily found from (28) to be

$$D_{ns} = \frac{1}{2!}\, \frac{d^2 j_n(k_{sn}R)}{dR^2}\, \frac{dP_n(\cos\Theta)}{d\Theta}\,, \quad F_{ns} = \frac{1}{2!}\, j_n(k_{sn}R)\, \frac{d^3 P_n(\cos\Theta)}{d\Theta^3} \tag{31}$$

The modes $\underset{\sim}{T}_{(ns)}$ are orthogonal so that if we write

$$\nabla \times \underset{\sim}{B} = \sum_{n,s} \underset{\sim}{T}_{(ns)}\,,$$

multiply by $\underset{\sim}{T}_{(pq)}$, and integrate over the volume of the core, we obtain

$$\iiint d^3\underset{\sim}{r}\, (\nabla \times \underset{\sim}{B}) \cdot \underset{\sim}{T}_{(pq)} = \iiint d^3\underset{\sim}{r}\, \underset{\sim}{T}_{(pq)} \cdot \underset{\sim}{T}_{(pq)} \tag{32}$$

Noting that

$$\int_0^\pi d\vartheta^* \sin\vartheta^* \left[\frac{dP_n(\cos\vartheta^*)}{d\vartheta^*}\right]^2 = \int_{-1}^{+1} d\mu\, (1-\mu^2)\left[P_n'(\mu)\right]^2 = \frac{2n(n+1)}{2n+1}$$

(MacRobert, 1948, pg. 105), and that

$$\int_0^P dr\, r^2\, j_n^2(k_{sn} r) = \frac{\pi}{2} \int_0^P dr\, r \left[J_{n+1/2}(k_{sn} r)\right]^2 = \frac{1}{2} P^3\, j_n(k_{sn} P)$$

(from (29) and Jahnke-Emde, 1945, pg. 146), we obtain

$$\iiint d^3\underset{\sim}{r}\, \underset{\sim}{T}_{(pq)} \cdot \underset{\sim}{T}_{(pq)} = c_{pq}^2\, P^3\, j_p^2(k_{qp} P)\, \pi\, \frac{2p(p+1)}{2p+1} \tag{33}$$

From (27) and (30) we find

$$\iiint d^3r\,(\nabla\times\mathbf{B})\cdot\mathbf{T}_{(pq)} = -\pi^{3/2} B_0\, c_{pq}\, a^4 \left[D_{pq} + \frac{F_{pq}}{R^2} + O\left(\frac{a}{R}\right)\right] \tag{34}$$

all other terms cancelling out for reasons of symmetry. Substituting (33) and (34) into (32) we obtain finally

$$c_{ns} = -B_0\left(D_{ns} + \frac{F_{ns}}{R^2}\right)\frac{\sqrt{\pi}\,a^4(2n+1)}{P^3 j_n^{\,2}(k_{sn}P)\,2n(n+1)} \tag{35}$$

We next consider the time dependence of the loops. They will appear in some random fashion and decay subsequently. The easiest way to discuss the decay of the normal modes will be to use the time dependent differential equation with a suitable source function $f_{ns}(t)$. The amplitude $\varkappa_{ns}(t)$ of the n,s mode then satisfies

$$\Lambda_{ns}\varkappa_{ns}(t) + \frac{d}{dt}\varkappa_{ns}(t) = f_{ns}(t), \quad \Lambda_{ns} = \frac{k_{sn}^2}{\mu\sigma} \tag{36}$$

The general solution of this equation is

$$\varkappa_{ns}(t) = \int_{-\infty}^{t} dt'\, f_{ns}(t')\exp\left[\Lambda_{ns}(t'-t)\right]$$

Let us assume that the loops appear suddenly at random times t_i; thereafter they merely spread out by diffusion. It is readily seen that the source function $f_{ns}(t)$ is proportional to the rate of appearance of the loops; we therefore take

$$f_{ns} = c_{ns}\sum_i \delta(t-t_i)$$

where $\delta(t-t_i)$ is a Dirac delta function. Then

$$\varkappa_{ns}(t) = c_{ns}\sum_i \exp\left[\Lambda_{ns}(t_i-t)\right]$$

where the summation is over all i for which $t > t_i$

The average of $\varkappa_{ns}(t)$ over time is

$$\langle\varkappa_{ns}\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{+T} dt\,\varkappa_{ns}(t)$$

which gives after some calculation

$$\langle \gamma_{ns} \rangle = c_{ns} \lim_{T\to\infty} \frac{1}{2T} \sum_i \frac{1}{\Lambda_{ns}} = \frac{\nu c_{ns}}{\Lambda_{ns}} \tag{37}$$

where ν is the number of loops produced per unit time.

We assume the apriori probability of a loop appearing at (R, Θ, Φ) to be $h(R)$, normalized to give unity when integrated over the sphere. The expectation value over both space and time of the amplitude γ_{ns} is

$$\langle\langle \gamma_{ns} \rangle\rangle = 2\pi \int_0^\pi d\Theta \sin\Theta \int_0^P dR\, R^2\, h(R) \langle \gamma_{ns} \rangle \tag{38}$$

Using (31), (35), and (37) we obtain

$$\langle\langle \gamma_{ns} \rangle\rangle = -\frac{(2n+1) B_0 \pi^{1/2} \nu \mu \sigma a^4}{2n(n+1)(k_{sn}P)^2 j_n^2(k_{sn}P)} \left[I_1(n) I_3(n,s) + I_2(n) I_4(n,s) \right] \tag{39}$$

and

$$I_1(n) = \int d\Theta \sin\Theta \frac{dP_n(\cos\Theta)}{d\Theta} \quad ; \quad I_2(n) = \int_0^\pi d\Theta \sin\Theta \frac{d^2 P_n(\cos\Theta)}{d\Theta^2}$$

$$I_3(n,s) = \int_0^{k_{sn}P} d(k_{sn}R)(k_{sn}R)^2 h(R) \frac{d^2 j_n(k_{sn}R)}{d(k_{sn}R)^2} \tag{40}$$

$$I_4(n,s) = \int_0^{k_{sn}P} d(k_{sn}R)\, h(R)\, j_n(k_{sn}R)$$

To evaluate $I_1(n)$ we integrate by parts and obtain

$$I_1(n) = -\int_0^\pi d\Theta \cos\Theta\, P_n(\cos\Theta)$$

This may be evaluated (MacRobert, 1948, pg. 106) to give

$$I_1(2n) = 0, \qquad I_1(2n+1) = \frac{\Gamma(n+\frac{1}{2})\Gamma(n+\frac{3}{2})}{\Gamma(n+1)\Gamma(n+2)} \tag{41}$$

To evaluate $I_2(n)$ we note that

$$\frac{dP_n(\cos\Theta)}{d\Theta} = -\sin\Theta \frac{dP_n(\cos\Theta)}{d(\cos\Theta)}$$

which vanishes at $\Theta = 0$ or π. Thus upon integration by parts we obtain

$$I_2(n) = - I_1(n) \tag{42}$$

To evaluate $I_3(n,s)$ and $I_4(n,s)$ we must assume a form for $h(R)$. Let us assume equal apriori probability of production of loops over a sphere of radius $R_0 \leq P$ and zero probability outside. With suitable normalization we then have

$$h(R) = \begin{cases} \frac{3}{4\pi R_0^3} & \text{for } R < R_0 \\ 0 & \text{for } R > R_0 \end{cases} \tag{43}$$

By (41) we need only consider modes of odd order, and for these,

$$\frac{4\pi R_0^3}{3} I_3(2n+1,s) = \int_0^{k_{s(2n+1)}R_0} du\, u^2 \frac{d^2 j_{2n+1}(u)}{du^2}, \quad \frac{4\pi R_0^3}{3} I_4(2n+1,s) = \int_0^{k_{s(2n+1)}R_0} du\, j_{2n+1}(u) \tag{44}$$

To evaluate $I_3(2n+1,s)$ we integrate by parts and obtain

$$I_3(2n+1,s) = 2 I_4(2n+1,s) + I_5(2n+1,s) \tag{45}$$

where

$$\frac{4\pi R_0^3}{3} I_5(2n+1,s) = (k_{s(2n+1)}R_0)^2 j_{2n}(k_{s(2n+1)}R_0) - 2(n+2)(k_{s(2n+1)}R_0) j_{2n+1}(k_{s(2n+1)}R_0) \tag{46}$$

Consider finally $I_4(2n+1,s)$. We use the identity

$$j_n(u) = - u^{n-1} \frac{d}{du}\left[u^{1-n} j_{n-1}(u)\right]$$

and integrate by parts. Repeating the process n times gives

$$\int du\, j_{2n+1}(u) = - \sum_{m=0}^{n} \frac{2^m n!}{(n-m)!} \frac{j_{2n-m}(u)}{u^m}$$

Noting that for small

$$j_n(u) \sim \frac{u^n}{(2n+1)!!}, \quad (2n+1)!! = (2n+1)(2n-1)\cdots 3\cdot 1$$

we have

$$\frac{4\pi R_0^3}{3} I_4(2n+1,s) = \frac{2^n n!}{(2n+1)!!} - \sum_{m=0}^{n} \frac{2^m n!}{(n-m)!} \frac{j_{2n-m}(k_{s(2n+1)}R_0)}{(k_{s(2n+1)}R_0)^m} \tag{47}$$

Using (42) and (45) we obtain

$$\langle\langle \gamma_{(2n+1)s} \rangle\rangle = - B\pi^{3/2} \nu\mu\sigma a^4 \frac{(4n+3) I_1(2n+1)\left[I_4(2n+1,s) + I_5(2n+1,s)\right]}{4(n+1)(2n+1)(k_{s(2n+1)}P)^3\, j_{2n+1}'(k_{s(2n+1)}P)} \tag{48}$$

Consider now the expansion of $\mathbf{B}$ in terms of orthogonal poloidal modes $\mathbf{S}_{(ns)}$. These are related to the $\mathbf{T}_{(ns)}$ by

$$\nabla \times \mathbf{T}_{(ns)} = \nabla \times (\nabla \times \mathbf{S}_{(ns)}) = -\nabla^2 \mathbf{S}_{(ns)} = k_{sn}^2 \mathbf{S}_{(ns)}$$

From (28) we compute then

$$S_{r(ns)} = c_{ns}\, n(n+1) P \frac{j_n(k_{sn} R) P_n(\cos\Theta)}{(k_{sn} P)(k_{sn} R)} \tag{49}$$

From (48), which gives $\langle\langle \gamma_{ns} \rangle\rangle$, the mean value of c_{ns}, and from (49) we find that the expectation value of B_r is

$$B_r = \sum_n \sum_s S_{r(ns)}$$

$$= P \sum_{n=0}^{\infty} (2n+1)(2n+2) \sum_{s=1}^{\infty} \langle\langle \gamma_{(2n+1)s} \rangle\rangle \frac{j_{2n+1}(k_{s(2n+1)} R) P_{2n+1}(\cos\Theta)}{(k_{s(2n+1)} P)(k_{s(2n+1)} R)}$$

$$= -\epsilon \sum_{n=0}^{\infty} (4n+3) I_1(2n+1) \sum_{s=1}^{\infty} \frac{(4\pi R_0^3/3)[I_4(2n+1,s) + I_5(2n+1,s)] j_{2n+1}(k_{s(2n+1)} R)}{(k_{s(2n+1)} P)^4 (k_{s(2n+1)} R)\, j_{2n+1}^2(k_{s(2n+1)} P)}$$

$$\times P_{2n+1}(\cos\Theta)$$

where

$$\epsilon = B_0 \frac{3\sqrt{\pi}\, \nu \mu \sigma a^4 P}{8 R_0^3} \tag{50}$$

The expectation value of the coefficient Γ_n of $P_n(\cos\Theta)$ is

$$\Gamma_{2n} = 0,\ \Gamma_{2n+1} = -\epsilon (4n+3) I_1(2n+1) \sum_{s=1}^{\infty} \frac{(4\pi R_0^3/3)[I_4(2n+1,s) + I_5(2n+1,s)] j_{2n+1}(k_{s(2n+1)} R)}{(k_{s(2n+1)} P)^4 (k_{s(2n+1)} R)\, j_{2n+1}^2(k_{s(2n+1)} P)} \tag{51}$$

To compare these results with observation we compute Γ_1 and Γ_3, the coefficients of the dipole and octupole moments for various values of R_0. The results are given in table 1 for the magnitude of these coefficients at the surface of the core, whence they can readily be converted to the corresponding values at the surface of the earth.

TABLE 1

THE COEFFICIENTS OF $P_1(\cos\Theta)$ AND $P_3(\cos\Theta)$ AT THE SURFACE OF THE CORE

$\frac{R_0}{P}$	$\frac{\Gamma_1(R_0)}{\epsilon}$	$\frac{\Gamma_3(R_0)}{\epsilon}$
0.5	-0.0222	-0.00086
0.8	-0.0816	-0.00235
0.9	-0.120	-0.01125
1.0	-0.162	+0.0251

Now we have shown previously that the loops produced by downgoing currents are opposite in sign to the loops produced by upgoing currents; we remarked that the loops produced by the downgoing currents were formed farther from the surface of the core than the loops formed by rising currents and therefore would make a much smaller contribution. Let us now show quantitatively that this is the case. Let us assume that the downgoing loops are produced only out to $R_0 = P/2$ and that they are opposite in sign and equal in number to the loops resulting from rising currents, which are produced in the region $P/2 < R < P$. Table 1 shows that $\Gamma_{1,3}(R_0) >> \Gamma_{1,3}(P/2)$ for $R_0 \cong P$.

The superposition of the upgoing and downgoing loops gives a net field of $\Gamma_{1,3}(R_0) - \Gamma_{1,3}(P/2)$. The ratio of the dipole to the octupole terms at $R = P$ is given in table 2 for $R_0 = 0.8, 0.9$ and 1.0. We expect that R_0 is somewhat less than P: It is physically impossible to produce a loop of finite size exactly at $R = P$; the toroidal field from which the loops are formed goes to zero at $R = P$ (Bullard, 1954). Observation indicates a value of about 15.7 for the ratio of the P_1 component to the P_3 component at the surface of the core, (Elsasser, 1941) which agrees with our calculations for $R_o \sim 0{\cdot}85\,P$.

TABLE 2.
RATIO OF ZONAL DIPOLE TO OCTUPOLE MODES
AT THE SURFACE OF THE CORE

$\frac{R_o}{P}$	$\frac{\Gamma_1(R_o) - \Gamma_1(P/2)}{\Gamma_3(R_o) - \Gamma_3(P/2)}$
0.8	18.5
0.9	8.1
1.0	5.4

There is probably no point in computing higher modes than the octupole term because the external fields of these modes are strongly influenced by the random motions near the surface of the core; thus, as with the observed quadrupole field, it is most likely that they arise mainly from random fluctuations rather than from the average feedback effect discussed here.

In the above numerical calculations we have neglected the effects of a possible inner solid core because the volume of such a sphere is small compared to the volume of the entire core, and because, as has been shown, $\Gamma_{2m+1}(R_o)$ decreases so rapidly with R_o. If one desired to take into account an inner core, it would only be necessary to superpose a third set of loops confined within a radius $R_1 \sim P/3$ to neutralize all other loops in $R < R_1$. The net field would then be $\Gamma_{2m+1}(R_o) - \Gamma_{2m+1}(P/2) + K\,\Gamma_{2m+1}(R_1)$ where K is equal to $(3/4\pi)[8/P^3 - 1/R_o^3]$. The effect of $K\,\Gamma_{2m+1}(R_1)$ will be small.

Finally, we note that, with a toroidal field which vanishes at the equator and the poles, our assumption that the loops are distributed uniformly over latitude is at best a rough approximation to the actual conditions; we may expect the loops to be concentrated in middle latitudes. If this is admitted, the ratio of the dipole to the octupole term may be made arbitrarily large, for instance by concentrating the loops near latitude $\pm 50^{\circ}$ where both P_3 and its second derivative are small.

6. Reversal of Field

Runcorn (1954) has cited geological evidence to the effect that the dipole field of the Earth has reversed itself in the past. The question naturally arises as to whether such a reversal can be incorporated into the model presented in this paper. Apparently there is a means of reversing the dipole if one assumes a sudden large increase in the convective motions within the core, as the following qualitative discussion will indicate.

The decay of the magnetic field in a dynamo may be thought of as a slipping of the lines of force back through the fluid, equivalent to an attempt to unwind the contortions introduced by the motion of the fluid. In a steady state dynamo it is physically obvious that the slip velocity and the fluid velocity are comparable, which is another way of saying that the characteristic time of the circulation of the fluid and the decay time of the magnetic field are comparable. Indeed, it can be shown that the regenerative operation of the dynamo requires a certain amount of phase shift resulting from the decay (Bondi and Gold, 1950); the decay acts in a very crude sense as the commutator of the dynamo. One might expect, then, that the regenerative process of the dynamo would be thrown out of gear if, for instance, the characteristic time of the circulation of the fluid were to be suddenly decreased.

To see just what will go wrong in our dynamo, let us consider a loop of flux produced near the surface of the core by a rising current as shown in figure 6. For normal steady operation of the dynamo, we find that during the time that the loop is being produced and then drawn out to reinforce the toroidal field, it will diffuse as is indicated by the broken line in figure 6; the side of the loop nearer the surface of the core will diffuse up through the surface of the core into the mantle and so escape the nonuniform rotation of the core. Hence it will not contribute to the toroidal field, which then results primarily from the side of the loop nearer the center of the core. But suppose, on the other hand, that just as the loop is being formed, the velocity of the fluid increases in order of magnitude. The diffusion of the loop will not have time to occur and both sides of the loop, not just the inner side, will be subject to the nonuniform rotation of the core and contribute to the toroidal field. The outer side of the loop is in a direction such that it will degenerate the existing toroidal field near the surface of the core. This degeneration will continue until the toroidal field reverses. The reversal will start at the surface of the core and occupy an increasingly thick shell; loops produced in the reversed layers will degenerate the dipole field. The characteristic decay time of the toroidal field is of the order of 10^4 years for $\sigma = 3 \times 10^5$ mho/m (Elsasser 1950), the characteristic time of formation of a loop is 600 years for a characteristic length of 1000 km and a velocity of 0.05 mm/sec. Thus, we see that the reversed toroidal field, if once begun by a brief outburst of convective motion, could persist long enough to completely degenerate and ultimately reverse the dipole field.

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Appendix

We shall treat the case of a convective stream closed in itself but so that the motion in one (say the upward) direction is strongly concentrated whereas the return flow is spread over a wide region. As announced in the text we shall show that the return flow makes only a negligible contribution to the formation of magnetic loop in the meridional planes.

We use the local cartesian system (ξ, η, ζ) and assume a toroidal field of the form (15). Let the velocity field of the fluid be $\underline{v} = \underline{u} + \underline{w}$: $\underline{u}$ represents the convection along the ζ-axis together with the associated influx and efflux and the necessary return flow, $\underline{w}$ represents the rotation of the fluid about the ζ-axis.

Without too much loss of generality we may take $u_\xi = 0$ and set

$$u_\eta = -u_0(t)\, X(\xi)\, Y(\eta)\, Z'(\zeta) \;, \quad u_\zeta = u_0(t)\, X(\xi)\, Y'(\eta)\, Z(\zeta)$$

where the primes denote derivatives. We represent $\underline{w}$ by $w_\zeta = 0$ and

$$w_\xi = -\Omega(t)\, \eta\, R(\rho)\, S(\zeta) \;, \quad w_\eta = \Omega(t)\, \xi\, R(\rho)\, S(\zeta)$$

where $u_0(t)$ and $\Omega(t)$ are representative of the corresponding magnitudes, provided all the other functions are suitably normalized. The forms (3) and (4) guarantee that $\nabla \cdot \underline{u}$ and $\nabla \cdot \underline{w}$ vanish.

From (1) we have for the first-order perturbation of the magnetic field,

$$\underline{\beta}(t) = \int_0^t dt \;\; \nabla \times \left[(\underline{u} + \underline{w}) \times \underline{B} \right]$$

The second-order perturbation field is

$$\underset{\sim}{\gamma}(t) = \int_0^t dt \; \nabla \times \left[(\underset{\sim}{u} + \underset{\sim}{w}) \times \underset{\sim}{B}(t) \right]$$

This field consists of the sum of four terms which arise depending on whether the first step of the interaction is taken to involve either $\underset{\sim}{u}$ or $\underset{\sim}{w}$, and whether the second step involves either $\underset{\sim}{u}$ or $\underset{\sim}{w}$ Two of these terms are quadratic, in $\underset{\sim}{u}$ alone and in $\underset{\sim}{w}$ alone; it is physically obvious that they will not contribute to loops in the meridional or $\zeta\xi$-planes. This may be demonstrated quantitatively by noting that the ξ-component of the term of second order in $u_o(t)$ and the ζ-component of the term of second order in $\Omega(t)$ are zero. Thus the terms cannot contribute individually. If we combine the nonzero ζ-component of the former and the nonzero ξ-component of the latter, we obtain a field which can be shown to have no net circulation about the η-axis. We therefore omit these quadratic terms and keep only the two mixed terms which involve $\underset{\sim}{u}$ in one step of the interaction and $\underset{\sim}{w}$ in the other. After somewhat lengthy but straightforward calculations this part of the second-order field, say $\underset{\sim}{\delta}(t)$, is found to have the following components in the meridional planes:

$$\begin{aligned} \delta_\xi(t) = f(t)X(\xi)\Big\{ &- \left[\frac{\partial}{\partial\eta} Y(\eta) \frac{\partial}{\partial\eta} \eta R(\rho)\right] Z'(\zeta)\, S(\zeta)\, B(\zeta) \\ &+ \left(\left[\frac{\partial}{\partial\eta} \eta R(\rho)\right] Y'(\eta) - R(\rho)\, \eta Y''(\eta) \right) \left[\frac{\partial}{\partial\zeta} Z(\zeta)\, S(\zeta)\, B(\zeta)\right] \\ &+ \left[\frac{\partial}{\partial\eta} Y'(\eta)\, \eta R(\rho)\right] S(\zeta) \left[\frac{\partial}{\partial\zeta} Z(\zeta)\, B(\zeta)\right] \Big\} \end{aligned} \qquad \text{(a)}$$

$$\begin{aligned} \delta_\zeta(t) = f(t)\Big\{ &- \left[\frac{\partial}{\partial\xi} X(\xi) \frac{\partial}{\partial\eta} \eta R(\rho)\right] Y'(\eta) + X(\xi) \left[\frac{\partial}{\partial\eta} \eta Y'(\eta) \frac{\partial}{\partial\xi} R(\rho)\right] \\ &+ \left[\frac{\partial}{\partial\xi} X(\xi) R(\rho)\right] \eta Y''(\eta) - \xi X(\xi) \left[\frac{\partial}{\partial\eta} R(\rho) Y''(\eta)\right] \Big\} Z(\zeta)\, S(\zeta)\, B(\zeta) \end{aligned} \qquad \text{(b)}$$

where

$$f(t) = \int_0^t dt' \, \upsilon_0(t') \int_0^{t'} dt'' \Omega(t'') = \int_0^t dt' \Omega(t') \int_0^{t'} dt'' \upsilon_0(t'')$$

The field lines of $\underset{\sim}{u}, \underset{\sim}{w}$ and $\underset{\sim}{\delta}$ are given by

$$\frac{d\eta}{u_\eta} = \frac{d\zeta}{u_\zeta} \quad , \quad \frac{d\xi}{w_\xi} = \frac{d\eta}{w_\eta} \quad , \quad \frac{d\xi}{\delta_\xi} = \frac{d\eta}{\delta_\eta} = \frac{d\zeta}{\delta_\zeta} \qquad \text{(c)}$$

For $\underset{\sim}{u}$ we obtain the family of curves

$$Y(\eta)\, Z(\zeta) = C_1 \qquad \text{(d)}$$

For $\underset{\sim}{w}$

$$\xi^2 + \eta^2 = C_2 \qquad \text{(e)}$$

where C_1 and C_2 are the parameters of each family. The differential equation for the field lines of $\underset{\sim}{\delta}$ is not readily integrated except in special cases. We therefore consider the case of a gaussian distribution

$$X(\xi) = \exp\left(-\frac{\xi^2}{a^2}\right) , \quad Y(\eta) = \eta \exp\left(-\frac{\eta^2}{a^2}\right) , \quad Z(\zeta) = \exp\left(-\frac{\zeta^2}{a^2}\right)$$

$$R(\rho) = \exp\left(-\frac{\rho^2}{a^2}\right) , \quad S(\zeta) = \exp\left(-\frac{\zeta^2}{a^2}\right) , \quad B(\zeta) = B_0 \exp\left[-\frac{(\zeta - b)^2}{c^2}\right]$$

Then (d) becomes

$$\eta \exp\left[-\frac{1}{a^2}(\eta^2 + \zeta^2)\right] = C_1 \quad \text{or} \quad \eta^2 + \zeta^2 = a^2 \ln \frac{\eta}{C_1} \qquad \text{(f)}$$

and (a) and (b) become

$$\delta_\xi(t) = f(t) B_0 2 \left[-\zeta \pi(\eta) + \nu(\eta)\right] \exp\left[-\frac{2}{a^2}(\xi^2 + \eta^2 + \zeta^2) - \frac{(\zeta - b)^2}{c^2}\right]$$

$$\delta_\zeta(t) = f(t) B_0 2 \xi \omega(\eta) \exp\left[-\frac{2}{a^2}(\xi^2 + \eta^2 + \zeta^2) - \frac{(\zeta - b)^2}{c^2}\right] \qquad \text{(g)}$$

where

$$\pi(\eta) = \frac{2}{a^2}\left(1 + 2\frac{\eta^2}{a^2}\right) + \frac{2}{c^2}\left(1 - 2\frac{\eta^2}{a^2}\right), \; \nu(\eta) = \frac{2b}{c^2}\left(1 - 2\frac{\eta^2}{a^2}\right), \; \omega(\eta) = \frac{4}{a^2}\left(1 - \frac{\eta^2}{a^2}\right) \qquad \text{(h)}$$

The neutral line where $\delta_\xi(t)$ and $\delta_\zeta(t)$ vanish simultaneously will be denoted by $\xi_0(\eta)$, $\zeta_0(\eta)$. We see from (g) that

$$\xi_0(\eta) = 0 , \quad \zeta_0(\eta) = \frac{\nu(\eta)}{\pi(\eta)} \tag{i}$$

Substituting (g) into (c) we obtain for the field lines of $\underset{\sim}{\mathfrak{b}}(t)$ the family of curves

$$\left[\zeta - \zeta_0(\eta)\right]^2 + \frac{\omega(\eta)}{\pi(\eta)}\,\xi^2 = C_3 \tag{k}$$

These are ellipses for $\eta^2 < a^2$ with centers on the neutral line. For $\eta^2 > a^2$ the lines form hyperbolas.

The problem of defining the "strength" of the magnetic loop generated offers some difficulty. The η-component of curl $\underset{\sim}{\mathfrak{b}}$ vanishes on integration over the volume, but this does not mean that there is no net circulation of magnetic flux about the neutral line. Perhaps the simplest way to demonstrate that such a circulation has been generated is the following: We compute the flux across the $\eta\zeta$-plane above $\zeta_0(\eta)$, and show that at least some of it bends around so as to penetrate the surface $\zeta = \zeta_0(\eta), \xi > 0$. The fluxes are

$$\Phi_\xi = +\int_{-\infty}^{+\infty} d\eta \int_{\zeta_0(\eta)}^{+\infty} d\zeta \;\; \mathfrak{b}_\xi \Big|^{\xi=0} \quad , \quad \Phi_\zeta = -\int_{-\infty}^{+\infty} d\eta \int_0^{\infty} d\xi \;\; \mathfrak{b}_\zeta \Big|^{\zeta=\zeta_0(\eta)}$$

Using (g) we find

$$\Phi_\xi = -2\, f(t) B_0 \int_{-\infty}^{+\infty} d\eta \int_{\zeta_0(\eta)}^{\infty} d\zeta \left[\zeta\,\pi(\eta) - \nu(\eta)\right] \exp\left[-\frac{2}{a^2}(\eta^2+\zeta^2) - \frac{1}{c^2}(\zeta-b)^2\right]$$

$$\Phi_\zeta = -2\, f(t) B_0 \int_{-\infty}^{+\infty} d\eta \;\; \omega(\eta) \exp\left[-\frac{2}{a^2}\left(\eta^2 + \zeta_0^2(\eta)\right) - \frac{1}{c^2}\left(\zeta_0(\eta) - b\right)^2\right]$$

$$\times \int_0^{\infty} d\xi \;\; \xi \exp\left(-\frac{2}{a^2}\xi^2\right)$$

In the integrand of the expression for Φ_ξ, we have $\zeta \geq \zeta_0(\eta)$ From (i) it follows that the integrand is always positive. In the expression for Φ_ζ the sign of the integrand is the same as the sign of $\omega(\eta)$. From (h) we see that $\omega(\eta)$ is positive if $\eta^2 < a^2$, otherwise negative. But for $\eta^2 > a^2$ the gaussian factor is very small, so small in fact that the integration over $\eta^2 > a^2$ constitutes only a few percent of the total value. Thus the integrals are positive and Φ_ξ and Φ_ζ have the same sign; hence there is a net circulation of flux about the neutral line independent of the relative magnitudes of a, b, and c.

Since the existence of a net circulation is independent of a, b, and c we shall limit the evaluation of the fluxes to the special case b = 0, a = c. Then $\zeta_0(\eta) = 0$ and

$$\Phi_\xi = \Phi_\zeta = -\frac{3\sqrt{\pi}}{2\sqrt{2}} f(t)\, B_0\, a$$

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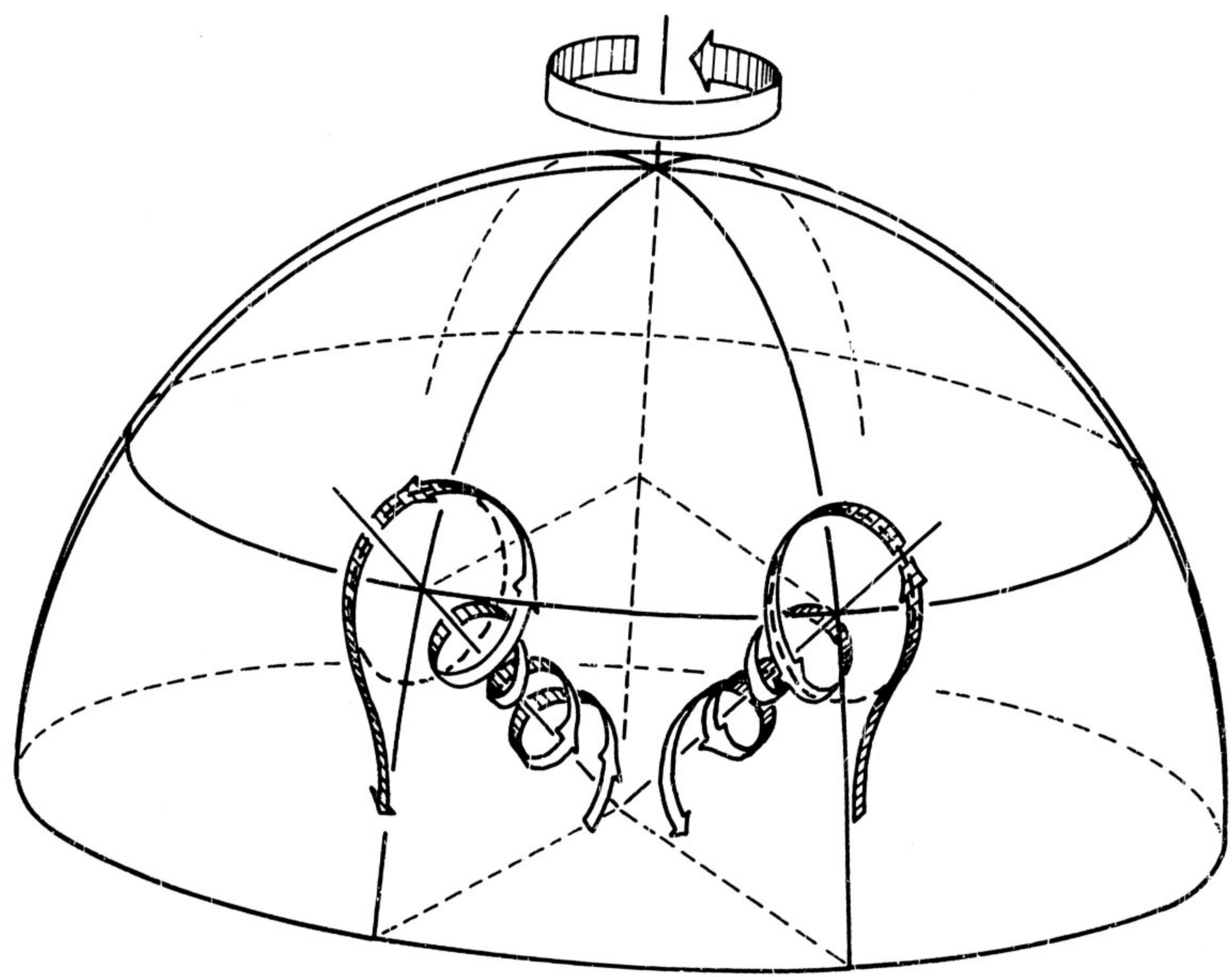
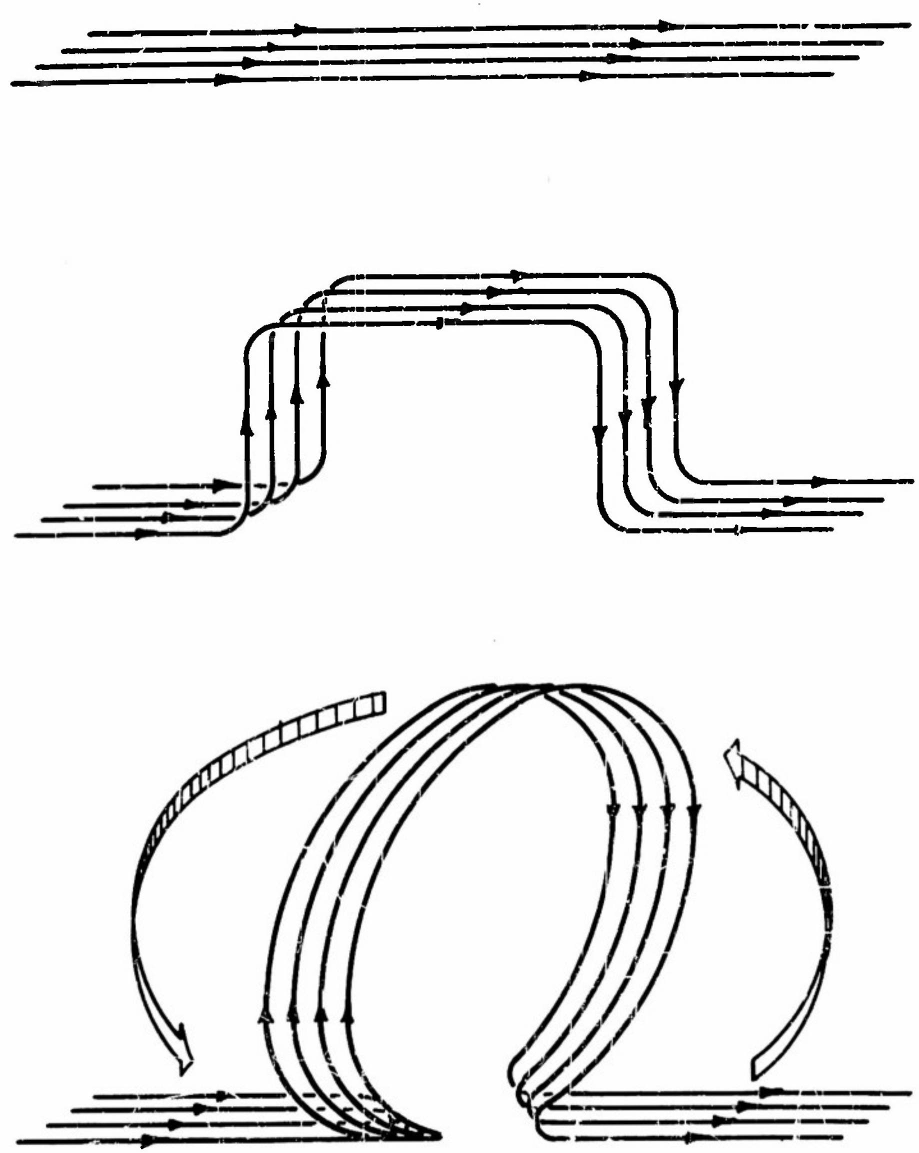


FIGURE 1



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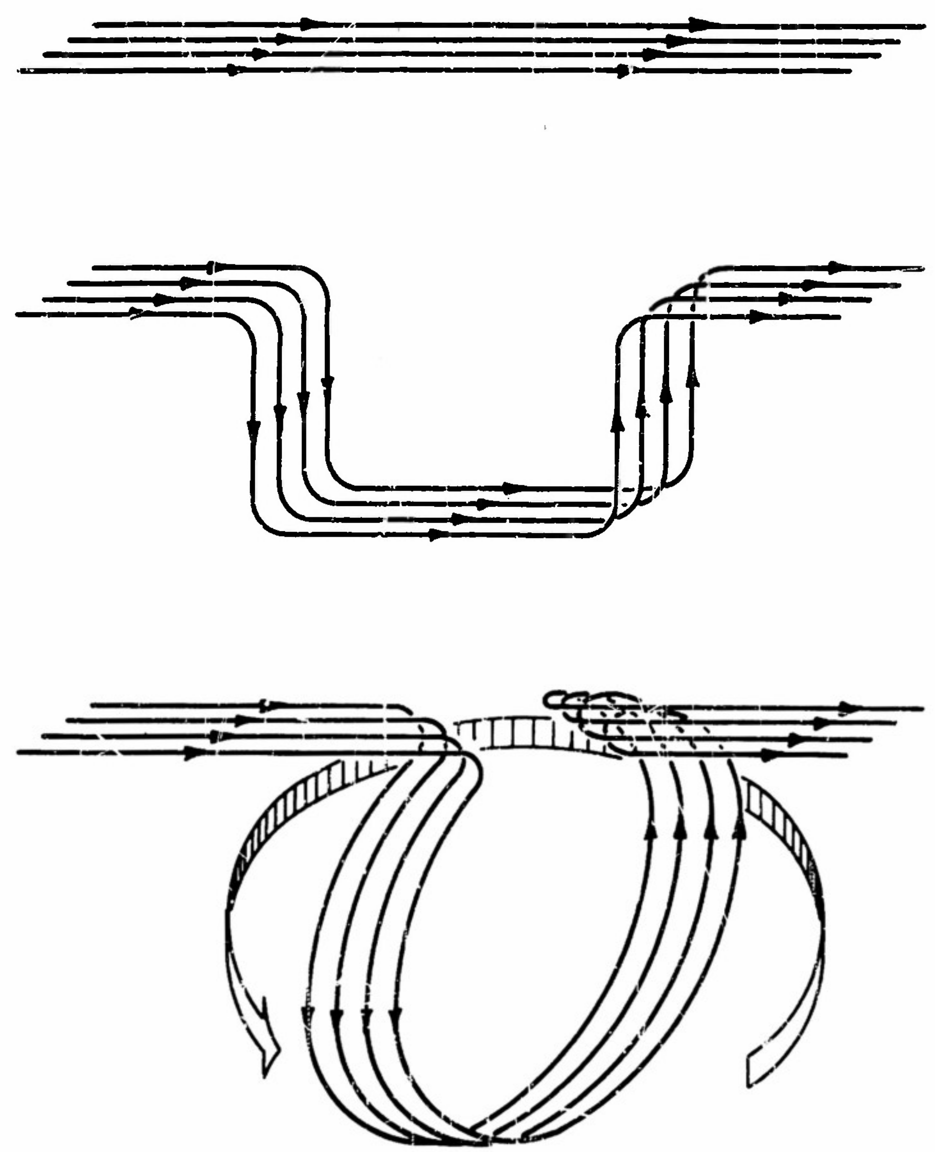

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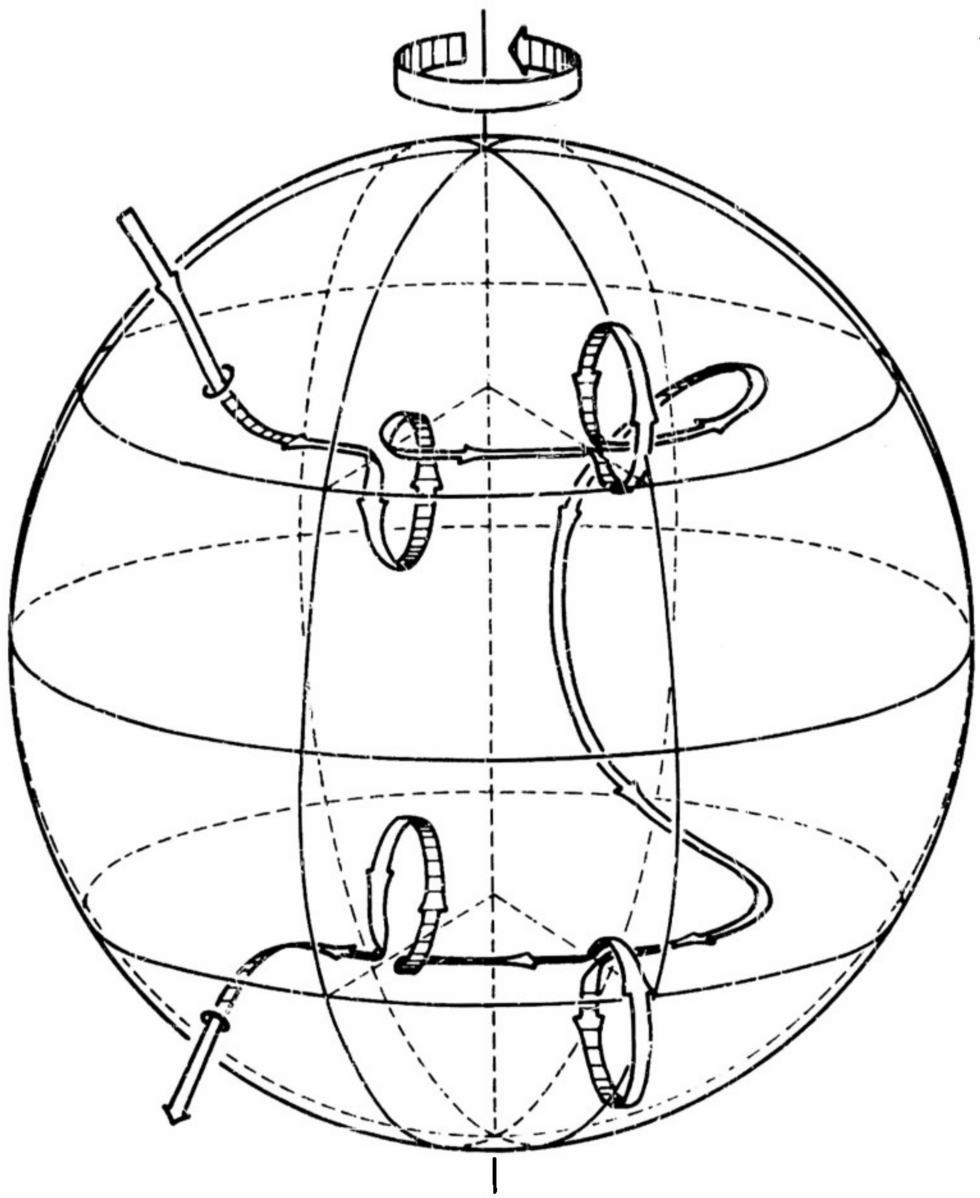

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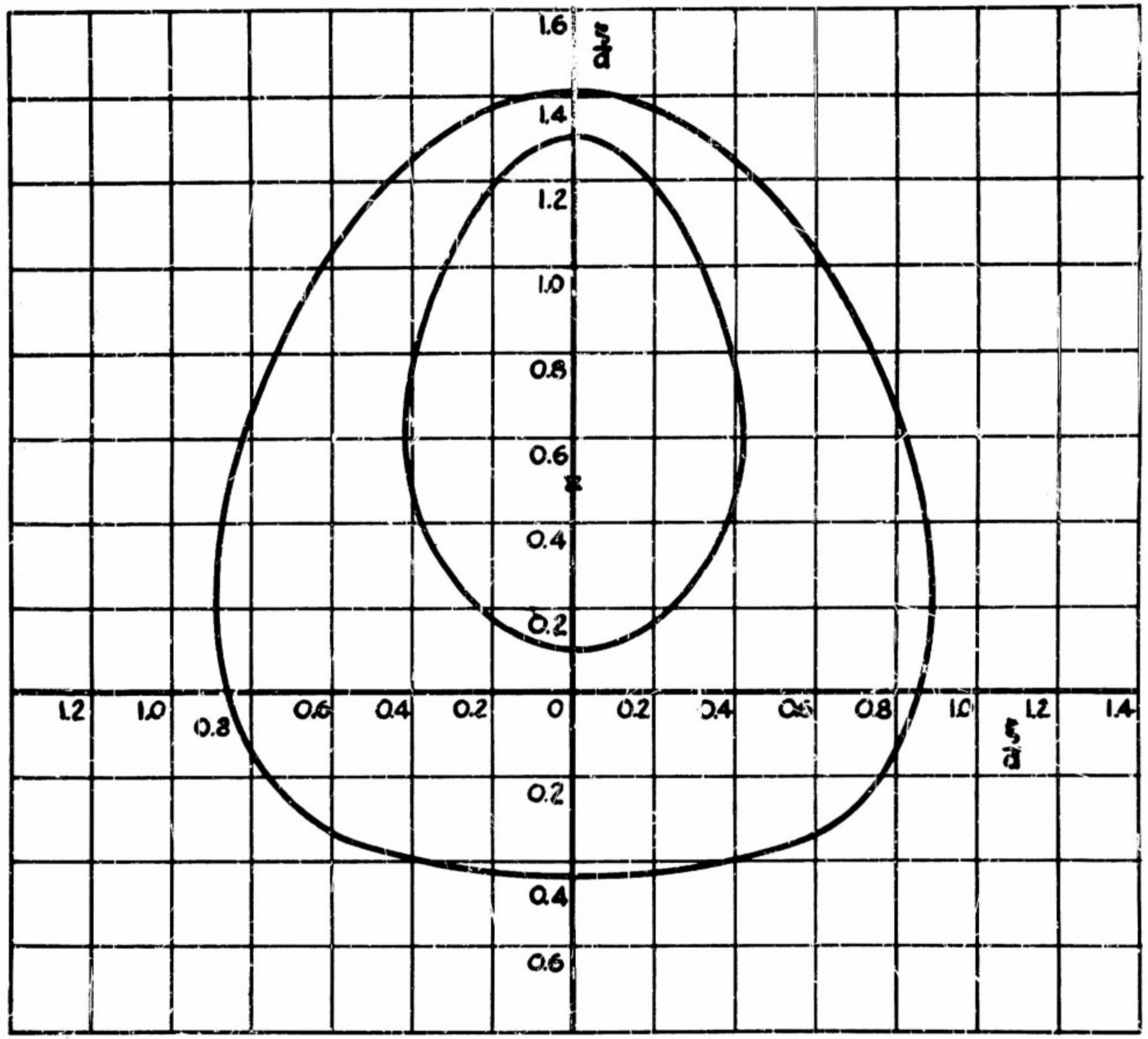

FIGURE 5

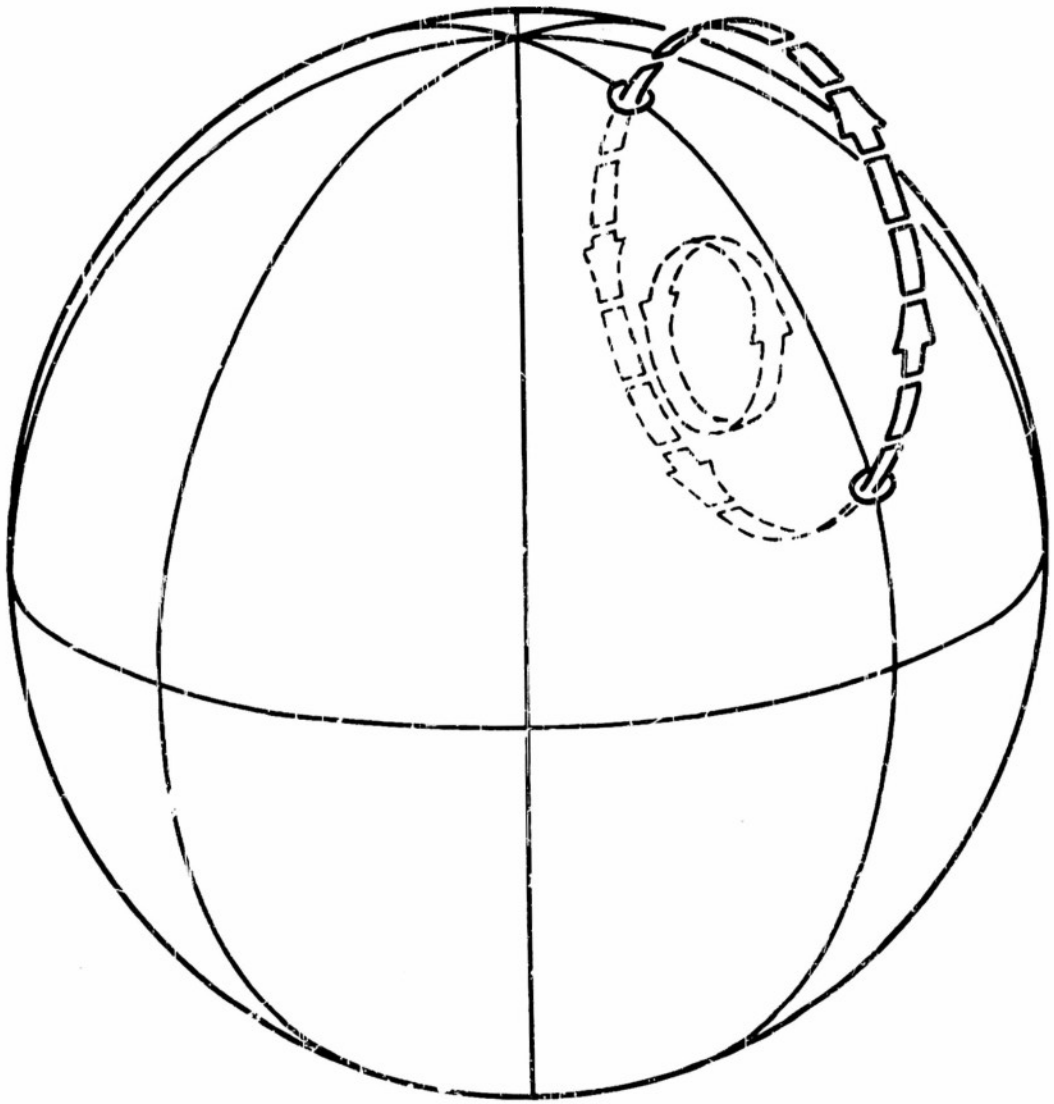

FIGURE 6

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